

微分方程式 II・自習シート

関数 $f(x)$ の $x = a$ における 1 次近似式と 2 次近似式はそれぞれ

$$f(a) + f'(a)(x - a),$$

$$f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$$

である.

問 1 次の関数 $f(x)$ の括弧内の点における 1 次近似式を求めよ.

(1) $f(x) = x^3 \quad (x = 2)$

(2) $f(x) = \sqrt{x} \quad (x = 4)$

(3) $f(x) = e^{2x} \quad (x = 0)$

(4) $f(x) = \frac{1}{1-x} \quad (x = 0)$

(5) $f(x) = \sin x \quad \left(x = \frac{\pi}{6}\right)$

(6) $f(x) = \log x \quad (x = e)$

(7) $f(x) = xe^x \quad (x = 1)$

(8) $f(x) = \tan x \quad \left(x = \frac{\pi}{4}\right)$

(9) $f(x) = e^{\sin x} \quad (x = \pi)$

(10) $f(x) = \log(x + \sqrt{x^2 + 1}) \quad (x = \sqrt{3})$

問 2 次の関数 $f(x)$ の括弧内の点における 2 次近似式を求めよ. 項を展開する必要は無い.

(1) $f(x) = x^4 \quad (x = 1)$

(2) $f(x) = e^x \quad (x = 0)$

(3) $f(x) = \cos x \quad (x = 0)$

(4) $f(x) = \sqrt{1+x} \quad (x = 0)$

(5) $f(x) = \frac{1}{x} \quad (x = 2)$

(6) $f(x) = \sqrt[3]{x} \quad (x = 1)$

(7) $f(x) = \sin x \quad \left(x = \frac{\pi}{3}\right)$

(8) $f(x) = x \log x \quad (x = e)$

(9) $f(x) = e^{-x^2} \quad (x = 1)$

(10) $f(x) = x^x \quad (x > 0) \quad (x = 1)$

問 1 の解答例 (1)

$$\begin{aligned}f(x) &= x^3, & f(2) &= 8, \\f'(x) &= 3x^2, & f'(2) &= 12.\end{aligned}$$

よって, $x = 2$ における 1 次近似式は

$$f(2) + f'(2)(x - 2) = 8 + 12(x - 2) = 12x - 16.$$

(2)

$$\begin{aligned}f(x) &= \sqrt{x} = x^{1/2}, & f(4) &= 2, \\f'(x) &= \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}, & f'(4) &= \frac{1}{4}.\end{aligned}$$

よって, $x = 4$ における 1 次近似式は

$$f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = \frac{1}{4}x + 1.$$

(3)

$$\begin{aligned}f(x) &= e^{2x}, & f(0) &= e^0 = 1, \\f'(x) &= 2e^{2x}, & f'(0) &= 2.\end{aligned}$$

よって, $x = 0$ における 1 次近似式は

$$f(0) + f'(0)(x - 0) = 1 + 2x.$$

(4)

$$\begin{aligned}f(x) &= \frac{1}{1-x} = (1-x)^{-1}, & f(0) &= 1, \\f'(x) &= (1-x)^{-2} = \frac{1}{(1-x)^2}, & f'(0) &= 1.\end{aligned}$$

よって, $x = 0$ における 1 次近似式は

$$f(0) + f'(0)(x - 0) = 1 + x.$$

(5)

$$\begin{aligned}f(x) &= \sin x, & f\left(\frac{\pi}{6}\right) &= \sin \frac{\pi}{6} = \frac{1}{2}, \\f'(x) &= \cos x, & f'\left(\frac{\pi}{6}\right) &= \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.\end{aligned}$$

よって, $x = \pi/6$ における 1 次近似式は

$$f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right).$$

(6)

$$\begin{aligned}f(x) &= \log x, & f(e) &= \log e = 1, \\f'(x) &= \frac{1}{x}, & f'(e) &= \frac{1}{e}.\end{aligned}$$

よって, $x = e$ における 1 次近似式は

$$f(e) + f'(e)(x - e) = 1 + \frac{1}{e}(x - e) = \frac{1}{e}x.$$

(7)

$$\begin{aligned} f(x) &= xe^x, & f(1) &= e, \\ f'(x) &= e^x + xe^x = (1+x)e^x, & f'(1) &= 2e. \end{aligned}$$

よって, $x = 1$ における 1 次近似式は

$$f(1) + f'(1)(x - 1) = e + 2e(x - 1) = 2ex - e.$$

(8)

$$\begin{aligned} f(x) &= \tan x, & f\left(\frac{\pi}{4}\right) &= \tan \frac{\pi}{4} = 1, \\ f'(x) &= \frac{1}{\cos^2 x}, & f'\left(\frac{\pi}{4}\right) &= \frac{1}{(1/\sqrt{2})^2} = 2. \end{aligned}$$

よって, $x = \pi/4$ における 1 次近似式は

$$f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) = 1 + 2\left(x - \frac{\pi}{4}\right).$$

(9)

$$\begin{aligned} f(x) &= e^{\sin x}, & f(\pi) &= e^{\sin \pi} = e^0 = 1, \\ f'(x) &= \cos x e^{\sin x}, & f'(\pi) &= \cos \pi e^0 = -1. \end{aligned}$$

よって, $x = \pi$ における 1 次近似式は

$$f(\pi) + f'(\pi)(x - \pi) = 1 - (x - \pi) = -x + 1 + \pi.$$

(10) $f(\sqrt{3}) = \log(\sqrt{3} + \sqrt{4}) = \log(2 + \sqrt{3})$. また合成関数の微分より

$$\begin{aligned} f'(x) &= \frac{1}{x + \sqrt{x^2 + 1}} \left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) \\ &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}} \\ &= \frac{1}{\sqrt{x^2 + 1}} \end{aligned}$$

であるから, $f'(\sqrt{3}) = \frac{1}{\sqrt{4}} = \frac{1}{2}$. よって, $x = \sqrt{3}$ における 1 次近似式は

$$f(\sqrt{3}) + f'(\sqrt{3})(x - \sqrt{3}) = \log(2 + \sqrt{3}) + \frac{1}{2}(x - \sqrt{3}).$$

問 2 の解答例 (1)

$$\begin{aligned}f(x) &= x^4, & f(1) &= 1, \\f'(x) &= 4x^3, & f'(1) &= 4, \\f''(x) &= 12x^2, & f''(1) &= 12.\end{aligned}$$

よって, $x = 1$ における 2 次近似式は

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 1 + 4(x-1) + 6(x-1)^2.$$

(2)

$$\begin{aligned}f(x) &= e^x, & f(0) &= 1, \\f'(x) &= e^x, & f'(0) &= 1, \\f''(x) &= e^x, & f''(0) &= 1.\end{aligned}$$

よって, $x = 0$ における 2 次近似式は

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + x + \frac{1}{2}x^2.$$

(3)

$$\begin{aligned}f(x) &= \cos x, & f(0) &= 1, \\f'(x) &= -\sin x, & f'(0) &= 0, \\f''(x) &= -\cos x, & f''(0) &= -1.\end{aligned}$$

よって, $x = 0$ における 2 次近似式は

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 - \frac{1}{2}x^2.$$

(4)

$$\begin{aligned}f(x) &= \sqrt{1+x} = (1+x)^{1/2}, & f(0) &= 1, \\f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f'(0) &= \frac{1}{2}, \\f''(x) &= -\frac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\frac{1}{4}.\end{aligned}$$

よって, $x = 0$ における 2 次近似式は

$$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

(5)

$$\begin{aligned}f(x) &= \frac{1}{x} = x^{-1}, & f(2) &= \frac{1}{2}, \\f'(x) &= -x^{-2}, & f'(2) &= -\frac{1}{4}, \\f''(x) &= 2x^{-3}, & f''(2) &= \frac{1}{4}.\end{aligned}$$

よって, $x = 2$ における 2 次近似式は

$$f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2.$$

(6)

$$\begin{aligned} f(x) &= \sqrt[3]{x} = x^{1/3}, & f(1) &= 1, \\ f'(x) &= \frac{1}{3}x^{-2/3}, & f'(1) &= \frac{1}{3} \cdot 1 = \frac{1}{3}, \\ f''(x) &= -\frac{2}{9}x^{-5/3}, & f''(1) &= -\frac{2}{9} \cdot 1 = -\frac{2}{9}. \end{aligned}$$

よって, $x = 1$ における 2 次近似式は

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = 2 + \frac{1}{3}(x-1) - \frac{1}{9}(x-1)^2.$$

(7)

$$\begin{aligned} f(x) &= \sin x, & f\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2}, \\ f'(x) &= \cos x, & f'\left(\frac{\pi}{3}\right) &= \frac{1}{2}, \\ f''(x) &= -\sin x, & f''\left(\frac{\pi}{3}\right) &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

よって, $x = \pi/3$ における 2 次近似式は

$$\frac{\sqrt{3}}{2} + \frac{1}{2}\left(x - \frac{\pi}{3}\right) - \frac{\sqrt{3}}{4}\left(x - \frac{\pi}{3}\right)^2.$$

(8)

$$\begin{aligned} f(x) &= x \log x, & f(e) &= e, \\ f'(x) &= \log x + 1, & f'(e) &= 2, \\ f''(x) &= \frac{1}{x}, & f''(e) &= \frac{1}{e}. \end{aligned}$$

よって, $x = e$ における 2 次近似式は

$$f(e) + f'(e)(x-e) + \frac{f''(e)}{2}(x-e)^2 = e + 2(x-e) + \frac{1}{2e}(x-e)^2.$$

(9)

$$\begin{aligned} f(x) &= e^{-x^2}, & f(1) &= \frac{1}{e}, \\ f'(x) &= -2xe^{-x^2}, & f'(1) &= -\frac{2}{e}, \\ f''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} = (4x^2 - 2)e^{-x^2}, & f''(1) &= \frac{2}{e}. \end{aligned}$$

よって, $x = 1$ における 2 次近似式は

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = \frac{1}{e} - \frac{2}{e}(x-1) + \frac{1}{e}(x-1)^2.$$

(10) $f(1) = 1^1 = 1$. $x > 0$ より両辺の対数をとると $\log f(x) = x \log x$ であるから, 両辺を x で微分して

$$\frac{f'(x)}{f(x)} = \log x + 1, \quad \text{すなわち} \quad f'(x) = x^x (\log x + 1).$$

よって $f'(1) = 1$. さらに積の微分より

$$\begin{aligned} f''(x) &= f'(x)(\log x + 1) + f(x) \cdot \frac{1}{x} \\ &= x^x (\log x + 1)^2 + x^{x-1} \end{aligned}$$

であるから, $f''(1) = 1 + 1 = 2$. よって, $x = 1$ における 2 次近似式は

$$f(1) + f'(1)(x - 1) + \frac{f''(1)}{2}(x - 1)^2 = 1 + (x - 1) + (x - 1)^2 = x^2 - x + 1.$$