微分方程式II・自習シート

問 1　次の問いに答えよ.

(1) $e > 2.70$ を証明せよ.

(2) I, B を

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

とする. I^2, I^3, B^2, B^3 をそれぞれ計算せよ. また, 2 次正方行列 A に対して

$$e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

で定義する. e^I, e^B がそれぞれどのような行列になるか計算せよ.

解答例　(1)

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

に注意する. $x = 1$ を代入すると

$$\begin{aligned} e &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \cdots \\ &> 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} \\ &= 2 + \frac{12 + 4 + 1}{24} \\ &> 2.70. \end{aligned}$$

(2)

$$I^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$I^3 = I^2 I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$B^2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B^3 = B^2 B = \begin{pmatrix} 2^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2^3 & 0 \\ 0 & 1 \end{pmatrix}.$$

提出する場合は, 解答例を参考にして自分で採点をしておくこと. 提出しなくても試験で 60 点以上取れば合格です.

よって,

$$e^I = \sum_{n=0}^{\infty} \frac{1}{n!} I^n = \begin{pmatrix} 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots & 0 \\ 0 & 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots \end{pmatrix}$$
$$= \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix},$$

$$e^B = \sum_{n=0}^{\infty} \frac{1}{n!} B^n = \begin{pmatrix} 1+2+\frac{1}{2!}2^2+\cdots+\frac{1}{n!}2^n+\cdots & 0 \\ 0 & 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots \end{pmatrix}$$
$$= \begin{pmatrix} e^2 & 0 \\ 0 & e \end{pmatrix}.$$

問 2　A, B を 2 次正方行列とする．すなわち

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

とする．フロベニウスノルム

$$\|A\| := \sqrt{\sum_{i,j=1}^{2} a_{ij}^2} \qquad \left(\text{つまり} = \sqrt{\sum_{i=1}^{2} \left(\sum_{j=1}^{2} a_{ij}^2 \right)} \quad \right)$$
$$= \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2} \quad (i, j = 1 \text{ から } i, j = 2 \text{ までのすべての和})$$

に対して

$$\|AB\| \leq \|A\| \|B\|$$

が成立することを証明せよ．ただし, つぎのシュワルツの不等式を用いてもよい．

$$ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \qquad (\text{全部ばらすならば})$$

$$\left(\sum_{k=1}^{2} c_k d_k \right)^2 \leq \left(\sum_{k=1}^{2} c_k^2 \right) \left(\sum_{k=1}^{2} d_k^2 \right) \qquad (\text{和の形を残しながらならば})$$

解答例　(その 1: 全部ばらしてひたすら計算する)

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

よって,

$$
\begin{aligned}
\|AB\| &= \left\{(a_{11}b_{11}+a_{12}b_{21})^2+(a_{11}b_{12}+a_{12}b_{22})^2+(a_{21}b_{11}+a_{22}b_{21})^2+(a_{21}b_{12}+a_{22}b_{22})^2\right\}^{\frac{1}{2}} \\
&= \left\{(a_{11}^2b_{11}^2+2a_{11}a_{12}b_{11}b_{21}+a_{12}^2b_{21}^2)+(a_{11}^2b_{12}^2+2a_{11}a_{12}b_{12}b_{22}+a_{12}^2b_{22}^2)\right. \\
&\quad \left.+(a_{21}^2b_{11}^2+2a_{21}a_{22}b_{11}b_{21}+a_{22}^2b_{21}^2)+(a_{21}^2b_{12}^2+2a_{21}a_{22}b_{12}b_{22}+a_{22}^2b_{22}^2)\right\}^{\frac{1}{2}} \\
&\leq \left\{\left(a_{11}^2b_{11}^2+2\left(\frac{1}{2}a_{11}^2b_{21}^2+\frac{1}{2}a_{12}^2b_{11}^2\right)+a_{12}^2b_{21}^2\right)\right. \\
&\quad +\left(a_{11}^2b_{12}^2+2\left(\frac{1}{2}a_{11}^2b_{22}^2+\frac{1}{2}a_{12}^2b_{12}^2\right)+a_{12}^2b_{22}^2\right) \\
&\quad +\left(a_{21}^2b_{11}^2+2\left(\frac{1}{2}a_{21}^2b_{21}^2+\frac{1}{2}a_{22}^2b_{11}^2\right)+a_{22}^2b_{21}^2\right) \\
&\quad \left.+\left(a_{21}^2b_{12}^2+2\left(\frac{1}{2}a_{21}^2b_{22}^2+\frac{1}{2}a_{22}^2b_{12}^2\right)+a_{22}^2b_{22}^2\right)\right\}^{\frac{1}{2}} \\
&= \left\{\left(a_{11}^2b_{11}^2+a_{11}^2b_{21}^2+a_{12}^2b_{11}^2+a_{12}^2b_{21}^2\right)+\left(a_{11}^2b_{12}^2+a_{11}^2b_{22}^2+a_{12}^2b_{12}^2+a_{12}^2b_{22}^2\right)\right. \\
&\quad \left.+\left(a_{21}^2b_{11}^2+a_{21}^2b_{21}^2+a_{22}^2b_{11}^2+a_{22}^2b_{21}^2\right)+\left(a_{21}^2b_{12}^2+a_{21}^2b_{22}^2+a_{22}^2b_{12}^2+a_{22}^2b_{22}^2\right)\right\}^{\frac{1}{2}} \\
&= \left\{\left(\sum_{i,j=1}^{2}a_{ij}^2\right)\left(\sum_{i,j=1}^{2}b_{ij}^2\right)\right\}^{\frac{1}{2}} \\
&= \|A\|\|B\|
\end{aligned}
$$

(その 2: 和の形を残しながら計算する)

$$
AB=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} \sum_{k=1}^{2}a_{1k}b_{k1} & \sum_{k=1}^{2}a_{1k}b_{k2} \\ \sum_{k=1}^{2}a_{2k}b_{k1} & \sum_{k=1}^{2}a_{2k}b_{k2} \end{pmatrix}
$$

よって,

$$
\begin{aligned}
\|AB\| &= \left\{ \left(\sum_{k=1}^{2} a_{1k} b_{k1} \right)^2 + \left(\sum_{k=1}^{2} a_{1k} b_{k2} \right)^2 + \left(\sum_{k=1}^{2} a_{2k} b_{k1} \right)^2 + \left(\sum_{k=1}^{2} a_{2k} b_{k2} \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 \right) + \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} b_{k2}^2 \right) \right. \\
&\quad \left. + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k2}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} (b_{k1}^2 + b_{k2}^2) \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 + \sum_{k=1}^{2} b_{k2}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} (a_{1k}^2 + a_{2k}^2) \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{i,k=1}^{2} a_{ik}^2 \right) \left(\sum_{i,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \|A\| \|B\|
\end{aligned}
$$