微分方程式II・自習シート

問 1　θ を実数とする．複素数 $i\theta$ について, 講義で証明した**オイラーの公式**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

を用いて数学に登場する重要な値「0,1,π,e,i」を結びつける次の等式

$$e^{i\pi} + 1 = 0$$

を証明せよ．

解答例　オイラーの公式より $\theta = \pi$ のとき

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0i = -1$$

よって -1 を移項して

$$e^{i\pi} + 1 = 0$$

を得る．

問 2　A, B を 2 次正方行列とする．すなわち

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

とする．フロベニウスノルム

$$\begin{aligned} \|A\| &:= \sqrt{\sum_{i,j=1}^{2} a_{ij}^2} \qquad \left(\text{つまり} = \sqrt{\sum_{i=1}^{2}\left(\sum_{j=1}^{2} a_{ij}^2\right)}\ \right) \\ &= \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2} \quad (i,j = 1 \text{ から } i,j = 2 \text{ までのすべての和}) \end{aligned}$$

に対して

$$\|AB\| \le \|A\|\|B\|$$

が成立することを証明せよ．ただし, つぎのシュワルツの不等式を用いてもよい．

$$ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2 \qquad (\text{全部ばらすならば})$$

$$\left(\sum_{k=1}^{2} c_k d_k\right)^2 \le \left(\sum_{k=1}^{2} c_k^2\right)\left(\sum_{k=1}^{2} d_k^2\right) \qquad (\text{和の形を残しながらならば})$$

提出する場合は, 解答例を参考にして自分で採点をしておくこと．提出しなくても試験で 60 点以上取れば合格です．

解答例 (その 1: 全部ばらしてひたすら計算する)

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

よって,

$$\begin{aligned}
\|AB\| &= \left\{(a_{11}b_{11} + a_{12}b_{21})^2 + (a_{11}b_{12} + a_{12}b_{22})^2 + (a_{21}b_{11} + a_{22}b_{21})^2 + (a_{21}b_{12} + a_{22}b_{22})^2\right\}^{\frac{1}{2}} \\
&= \left\{(a_{11}^2b_{11}^2 + 2a_{11}a_{12}b_{11}b_{21} + a_{12}^2b_{21}^2) + (a_{11}^2b_{12}^2 + 2a_{11}a_{12}b_{12}b_{22} + a_{12}^2b_{22}^2)\right. \\
&\quad \left. + (a_{21}^2b_{11}^2 + 2a_{21}a_{22}b_{11}b_{21} + a_{22}^2b_{21}^2) + (a_{21}^2b_{12}^2 + 2a_{21}a_{22}b_{12}b_{22} + a_{22}^2b_{22}^2)\right\}^{\frac{1}{2}} \\
&\leq \left\{\left(a_{11}^2b_{11}^2 + 2\left(\frac{1}{2}a_{11}^2b_{21}^2 + \frac{1}{2}a_{12}^2b_{11}^2\right) + a_{12}^2b_{21}^2\right)\right. \\
&\quad + \left(a_{11}^2b_{12}^2 + 2\left(\frac{1}{2}a_{11}^2b_{22}^2 + \frac{1}{2}a_{12}^2b_{12}^2\right) + a_{12}^2b_{22}^2\right) \\
&\quad + \left(a_{21}^2b_{11}^2 + 2\left(\frac{1}{2}a_{21}^2b_{21}^2 + \frac{1}{2}a_{22}^2b_{11}^2\right) + a_{22}^2b_{21}^2\right) \\
&\quad \left. + \left(a_{21}^2b_{12}^2 + 2\left(\frac{1}{2}a_{21}^2b_{22}^2 + \frac{1}{2}a_{22}^2b_{12}^2\right) + a_{22}^2b_{22}^2\right)\right\}^{\frac{1}{2}} \\
&= \left\{\left(a_{11}^2b_{11}^2 + a_{11}^2b_{21}^2 + a_{12}^2b_{11}^2 + a_{12}^2b_{21}^2\right) + \left(a_{11}^2b_{12}^2 + a_{11}^2b_{22}^2 + a_{12}^2b_{12}^2 + a_{12}^2b_{22}^2\right)\right. \\
&\quad \left. + \left(a_{21}^2b_{11}^2 + a_{21}^2b_{21}^2 + a_{22}^2b_{11}^2 + a_{22}^2b_{21}^2\right) + \left(a_{21}^2b_{12}^2 + a_{21}^2b_{22}^2 + a_{22}^2b_{12}^2 + a_{22}^2b_{22}^2\right)\right\}^{\frac{1}{2}} \\
&= \left\{\left(\sum_{i,j=1}^{2} a_{ij}^2\right)\left(\sum_{i,j=1}^{2} b_{ij}^2\right)\right\}^{\frac{1}{2}} \\
&= \|A\|\|B\|
\end{aligned}$$

(その 2: 和の形を残しながら計算する)

$$AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{2} a_{1k}b_{k1} & \sum_{k=1}^{2} a_{1k}b_{k2} \\ \sum_{k=1}^{2} a_{2k}b_{k1} & \sum_{k=1}^{2} a_{2k}b_{k2} \end{pmatrix}$$

よって,

$$
\begin{aligned}
\|AB\| &= \left\{ \left(\sum_{k=1}^{2} a_{1k} b_{k1} \right)^2 + \left(\sum_{k=1}^{2} a_{1k} b_{k2} \right)^2 + \left(\sum_{k=1}^{2} a_{2k} b_{k1} \right)^2 + \left(\sum_{k=1}^{2} a_{2k} b_{k2} \right)^2 \right\}^{\frac{1}{2}} \\
&\leq \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 \right) + \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} b_{k2}^2 \right) \right. \\
&\quad \left. + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k2}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k=1}^{2} (b_{k1}^2 + b_{k2}^2) \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k=1}^{2} b_{k1}^2 + \sum_{k=1}^{2} b_{k2}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} a_{1k}^2 \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) + \left(\sum_{k=1}^{2} a_{2k}^2 \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{k=1}^{2} (a_{1k}^2 + a_{2k}^2) \right) \left(\sum_{k,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \left\{ \left(\sum_{i,k=1}^{2} a_{ik}^2 \right) \left(\sum_{i,j=1}^{2} b_{kj}^2 \right) \right\}^{\frac{1}{2}} \\
&= \|A\| \|B\|
\end{aligned}
$$